REFORMULATION THE YANG-MILLS FIELD BY FRACTIONAL CALCULUS

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ABSTRACT: *In this paper we reformulate the Yang-Mills field using fractional calculus with left-right Riemann-Liouville fractional derivatives. Yang-Mills field is also rewritten using the technique of fractional variation principle. Euler-Lagrange equations and Hamiltonian equations of Yang-Mills field are then determined by the two definitions: fractional calculus and variation principle. It is found that the classical results are obtained as a particular case of the fractional formulation for Euler-Lagrange and Hamiltonian equations*.

Keywords: Yang-Mills Field; Fractional Derivatives; Fractional Euler-Lagrange Equation; Lagrangian and Hamiltonian Formulations.

1. INTRODUCTION

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders (including complex orders), and their applications in science, engineering, mathematics, economics, and other fields are based on replacing the time derivative in an evolution equation with a derivative of fractional order [1, 2, 3, 4, 5, 6, 7, 8, 9]. The results of several recent researchers confirm that fractional derivatives seem to arise for important mathematical reasons.

The fractional variational principle represents an important part of fractional calculus. Riewe developed fractional Lagrangian, fractional Hamiltonian, and fractional mechanics [10,11]. He has shown that Lagrangian with fractional derivative lead directly to equations of motion with non-conservative classical forces such as friction. Agrawal [12, 13] has presented Euler-Lagrange equations for unconstrained and constrained fractional variational problems and he developed a formulation of Euler-Lagrange equations for continuous systems, he also presented the transversality condition for fractional variational problems. Baleanu et al [14,15,16,17] developed fractional Lagrangian and Hamiltonian of mechanical and field systems. They investigated Euler-Lagrange equations and the fractional Hamilton equations corresponding to a fractional generalization of the equivalent Lagrangians. Rabei et al [18] investigated the classical field with fractional derivatives using the Hamiltonian formalism for discrete and continuous systems.

As a new application, in this paper we reformulate the Yang-Mills Lagrangian density in fractional form in terms of the Riemann–Liouville fractional derivative and obtain the equations of motion and compare them with the Hamilton's equations of motion in fractional form. It is well known that the Yang-Mills Lagrangian density is a scalar function that describes all the mass and energy interaction per unit volume. It also describes the charge of the particle or particles and the electromagnetic interacting charges with the field.

In the following, mathematical tools are briefly reviewed. Then in Sec. 3 we present a new fractional Yang-Mills Lagrangian density. Then using the fractional Euler-Lagrange equations we obtain fractional Yang-Mills equations. After that we construct the fractional Hamiltonian equations within Riemann – Liouville fractional derivative from fractional Yang-Mills Lagrangian density. At last, in Sec. 4, we will present some conclusions.

2. Mathematical Tools

Here we give the standard definitions of left and right Riemann-Liouville fractional derivatives and Caputo fractional derivatives.

The left Riemann-Liouvilla fractional derivative, which is denoted by LRLFD, reads as [19]

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(t)d\tau \quad (1)$$

And the form of right Riemann-Liouville fractional derivative, which is denoted by RRLFD, is given below

$$ {}_tD_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{-d}{dt}\right)^n \int_t^a (t-\tau)^{n-\alpha-1} f(t)d\tau \quad (2)$$

Here α is the order of derivative such that (n-1<α≤n) and is not equal zero. If α is and integer, these derivatives become the usual derivatives.

$$ {}_aD_t^\alpha f(t) = \left(\frac{d}{dt}\right)^\alpha f(t) \quad (3)$$

$$ {}_tD_a^\alpha f(t)=\left(\frac{-d}{dt}\right)^\alpha f(t) \quad ;\alpha=1, 2, 3\ldots \quad (4).$$

The left Caputo fractional derivative which is denoted by LCFD reads as [13]

$$ {}_t^cD_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f(t)d\tau \quad (5)$$

And right Caputo fractional derivatives which is denoted by RCFD reads as

$$ {}_t^cD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (t-\tau)^{n-\alpha-1}\left(-\frac{d}{dt}\right)^n f(t)d\tau \quad (6)$$

Here (Γ) represents the gamma function. The Riemann-Liouville derivative of constant is not zero, although Caputo derivative of a constant is zero.

3. 3. Fractional Treatment

3.1 Fractional Yang-Mills Lagrangian Density

The concept of Lagrangian density is defined as the Kinetic energy per unit volume minus the potential energy per unit volume. The usual Lagrangian of the system equals the Lagrangian density integrated over the whole volume of the system.

The Lagrangian density for Yang-Mills field must be a scalar invariant constructed from the field tensor, the four vectors, and the currents. This means that all scalar products must be constructed from two quantities.

$$ L= -\frac{1}{4}\, g_{\mu\lambda}g_{\nu\beta}F^{\lambda\beta}g^{\mu\sigma}g^{\nu\alpha}F^{\sigma\alpha} + i\gamma^\mu\bar{\Psi}\partial_\mu\Psi - e\gamma^\mu A_\mu\Psi\bar{\Psi} - m\Psi\bar{\Psi} \quad (7)$$

To rewrite the Yang -Mills Lagrangian density in Riemann – Liouville fractional form use these relations

$$F_{\mu v} = \left({}_aD^{\alpha}_{x_\mu} A_v - {}_aD^{\alpha}_{x_v} A_\mu \right) \quad (8)$$

$$F^{\mu v} = \left({}_aD^{\alpha}_{x^\mu} A^v - {}_aD^{\alpha}_{x^v} A^\mu \right) \quad (9)$$

$$F_{\mu v}F^{\mu v} = 2\left[{}_aD^{\alpha}_{x_\mu} A_v \, {}_aD^{\alpha}_{x^\mu} A^v - {}_aD^{\alpha}_{x_v} A_\mu \, {}_aD^{\alpha}_{x^v} A^\mu \right] \quad (10)$$

Where $F_{\mu v}$ or $F^{\mu v}$is a four dimension antisymmetric second rank tensor, $\mu = 0\,, i\,; i = 1,2,3$ and $v = 0\,,\ j\,; j =$ 1,2,3.

Then the fractional Yang -Mills Lagrangian density become s

$$\mathrm{L} = -\frac{1}{2}\begin{bmatrix} \left({}_aD^{\alpha}_t A_j\right)\left({}_aD^{\alpha}_t A^j\right) - \left({}_aD^{\alpha}_t A_j\right)\left({}_aD^{\alpha}_{x^j} \emptyset\right) + \\ \left({}_aD^{\alpha}_{x_i} A_0\right)\left({}_aD^{\alpha}_{x^i} A_0\right) - \left({}_aD^{\alpha}_{x_i} A_0\right)\left({}_aD^{\alpha}_t A^i\right) + \\ \left({}_aD^{\alpha}_{x_i} A_j\right)\left({}_aD^{\alpha}_{x^i} A^j\right) - \left({}_aD^{\alpha}_{x_i} A_j\right)\left({}_aD^{\alpha}_{x^j} A^i\right) \end{bmatrix} + \begin{bmatrix} i\gamma^0 \overline{\Psi} \, {}_aD^{\alpha}_t \Psi \\ + i\gamma^i \overline{\Psi} \, {}_aD^{\alpha}_{x_i} \Psi \\ -e\gamma^0 \phi \Psi \overline{\Psi} - \\ e\gamma^i A_i \Psi \overline{\Psi} - m \Psi \overline{\Psi} \end{bmatrix} \quad (11)$$

Use these relativistic notations

$$A^{\alpha} = \left(\emptyset\,, \vec{A}\right)$$
$$A_{\alpha} = \left(\emptyset\,, -\vec{A}\right)$$
$$\partial_{\alpha} = {}_aD^{\alpha}_{x_\mu} = \left({}_aD^{\alpha}_t\,, {}_aD^{\alpha}_{x_j}\right)$$
$$\partial^{\alpha} = {}_aD^{\alpha}_{x^\mu} = \left({}_aD^{\alpha}_t, -{}_aD^{\alpha}_{x^j}\right)$$

Thus, the fractional Yang-Mills Lagrangian density in terms of the Riemann – Liouville fractional derivative becomes:

$$\mathcal{L} = -\frac{1}{2}\begin{bmatrix} -{}_aD^{\alpha}_t A^j \, {}_aD^{\alpha}_t A^j + {}_aD^{\alpha}_t A^j \, {}_aD^{\alpha}_{x^j} \emptyset \\ -{}_aD^{\alpha}_{x^i} \emptyset \, {}_aD^{\alpha}_{x^i} \emptyset + {}_aD^{\alpha}_{x^i} \emptyset \, {}_aD^{\alpha}_i A^i + \\ {}_aD^{\alpha}_{x^i} A^j \, {}_aD^{\alpha}_{x^i} A^j - {}_aD^{\alpha}_{x^i} A^j \, {}_aD^{\alpha}_{x^j} A^i \end{bmatrix} + \begin{bmatrix} i\gamma^0 \overline{\Psi} \, {}_aD^{\alpha}_t \Psi \\ + i\gamma^i \overline{\Psi} \, {}_aD^{\alpha}_{x_i} \Psi \\ -e\gamma^i A_i \Psi \overline{\Psi} \\ -e\gamma^0 \phi \Psi \overline{\Psi} - m \Psi \overline{\Psi} \end{bmatrix} \quad (12)$$

Similarly, we can use the left Caputo fractional derivative LCFD definition to rewrite the fractional Yang-Mills Lagrangian density**,** where the stress tensor takes the form in LCFD**:**

$$F_{\mu v} = \left({}^c_aD^{\alpha}_{x_\mu} A_v - {}_aD^{\alpha}_{x_v} A_\mu \right)$$
$$F^{\mu v} = \left({}^c_aD^{\alpha}_{x^\mu} A^v - {}^c_aD^{\alpha}_{x^v} A^\mu \right)$$
$$F_{\mu v}F^{\mu v} = 2\left[{}^c_aD^{\alpha}_{x_\mu} A_v \, {}^c_aD^{\alpha}_{x^\mu} A^v - {}^c_aD^{\alpha}_{x_v} A_\mu \, {}^c_aD^{\alpha}_{x^v} A^\mu \right] \quad (13)$$

$$\mathcal{L} = -\frac{1}{2}\begin{bmatrix} - {}^c_aD^{\alpha}_t A^j \, {}^c_aD^{\alpha}_t A^j + {}^c_aD^{\alpha}_t A^j \, {}^c_aD^{\alpha}_{x^j} \emptyset \\ - {}^c_aD^{\alpha}_{x^i} \emptyset \, {}^c_aD^{\alpha}_{x^i} \emptyset + {}^c_aD^{\alpha}_{x^i} \emptyset \, {}^c_aD^{\alpha}_t A^i + \\ {}^c_aD^{\alpha}_{x^i} A^j \, {}^c_aD^{\alpha}_{x^i} A^j - {}^c_aD^{\alpha}_{x^i} A^j \, {}^c_aD^{\alpha}_{x^j} A^i \end{bmatrix} + \begin{bmatrix} i\gamma^0 \overline{\Psi} \, {}^c_aD^{\alpha}_t \Psi \\ + i\gamma^i \overline{\Psi} \, {}^c_aD^{\alpha}_i \Psi \\ -e\gamma^0 A_i \Psi \overline{\Psi} \\ -e\gamma^0 \phi \Psi \overline{\Psi} - m \Psi \overline{\Psi} \end{bmatrix} \quad (14)$$

3.2 Fractional Euler – Lagrange Equation for Yang-Mills Lagrangian Density

Consider the action function of the form

$$j = \int \mathrm{L}\left(\Psi(x), \overline{\Psi},\ A_\mu\,, {}_a\mathrm{D}^{\gamma}_{x_\rho} \Psi, {}_a\mathrm{D}^{\gamma}_{x_\rho} \overline{\Psi},\ {}_a\mathrm{D}^{\gamma}_{x_\rho} A_\mu \right) d^3x dt = 0 \quad (15)$$

$$\delta j = \int \delta\, \mathrm{L}\left(\Psi(x), \overline{\Psi},\ A_\mu\,, {}_a\mathrm{D}^{\gamma}_{x_\rho} \Psi, {}_a\mathrm{D}^{\gamma}_{x_\rho} \overline{\Psi},\ {}_a\mathrm{D}^{\gamma}_{x_\rho} A_\mu \right) d^3x dt = 0 \quad (16)$$

Or

$$\delta j = \int \left\{ \begin{matrix} \frac{\partial \mathrm{L}}{\partial \Psi} \delta \Psi + \frac{\partial L}{\partial \overline{\Psi}} \delta \overline{\Psi} + \frac{\partial L}{\partial A_\mu} \delta A_\mu \\ + \frac{\partial L}{\partial\left({}_{x_\rho}D^{\gamma}_a A_\mu\right)} \delta\left({}_{x_\rho}D^{\gamma}_a A_\mu\right) \\ \frac{\partial \mathrm{L}}{\partial\left({}_aD^{\gamma}_{x_\rho} A_\mu\right)} \delta\left({}_aD^{\gamma}_{x_\rho} A_\mu\right) + \\ \frac{\partial \mathrm{L}}{\partial\left({}_{x_\rho}D^{\gamma}_a \Psi\right)} \delta\left({}_{x_\rho}D^{\gamma}_a \Psi\right) + \\ + \frac{\partial \mathrm{L}}{\partial\left({}_aD^{\gamma}_{x_\rho} \overline{\Psi}\right)} \delta\left({}_aD^{\gamma}_{x_\rho} \overline{\Psi}\right) + \\ \frac{\partial \mathrm{L}}{\partial\left({}_{x_\rho}D^{\gamma}_a \overline{\Psi}\right)} \delta\left({}_{x_\rho}D^{\gamma}_a \overline{\Psi}\right) \\ + \frac{\partial \mathrm{L}}{\partial\left({}_aD^{\gamma}_{x_\rho} \Psi\right)} \delta\left({}_aD^{\gamma}_{x_\rho} \Psi\right) \end{matrix} \right\} d^4x = 0 \quad (17)$$

Using

$\delta\left({}_aD^{\gamma}_{x_\rho} \Psi\right) = {}_aD^{\gamma}_{x_\rho} \delta \Psi$ and $\delta\left({}_aD^{\gamma}_{x_\rho} \overline{\Psi}\right) = {}_aD^{\gamma}_{x_\rho} \delta \overline{\Psi}$

$\delta\left({}_aD^{\gamma}_{x_\rho} A_\mu\right) = {}_aD^{\gamma}_{x_\rho} \delta A_\mu$ and $\delta\left({}_{x_\rho}D^{\gamma}_a \Psi\right) = {}_{x_\rho}D^{\gamma}_a \delta \Psi$

$\delta\left({}_{x_\rho}D^{\gamma}_a \overline{\Psi}\right) = {}_{x_\rho}D^{\gamma}_a \delta \overline{\Psi}$ and $\delta\left({}_{x_\rho}D^{\gamma}_a A_\mu\right) = {}_{x_\rho}D^{\gamma}_a \delta A_\mu$

We obtain

$$\delta j = \int \left\{ \begin{array}{c} \frac{\partial L}{\partial \Psi}\delta\Psi + \frac{\partial L}{\partial \bar{\Psi}}\delta\bar{\Psi} + \\ \frac{\partial L}{\partial A_\mu}\delta A_\mu + \underbrace{\frac{\partial L}{\partial ({}_aD^{\gamma}_{x_\rho}\Psi)}\, {}_aD^{\gamma}_{x_\rho}\delta\Psi}_{fourthhhh} \\ + \underbrace{\frac{\partial L}{\partial ({}_{x_\rho}D^{\beta}_{a}\Psi)}\, {}_{x_\rho}D^{\beta}_{a}\delta\Psi}_{fifth} + \underbrace{\frac{\partial L}{\partial ({}_{x_\rho}D^{\beta}_{a}A_\mu)}\, {}_aD^{\beta}_{x_\rho}\delta A_\mu}_{sixthh} \\ + \underbrace{\frac{\partial L}{\partial ({}_{x_\rho}D^{\beta}_{a}\bar{\Psi})}\, {}_{x_\rho}D^{\beta}_{a}\delta\bar{\Psi}}_{seventhh} + \underbrace{\frac{\partial L}{\partial ({}_aD^{\gamma}_{x_\rho}\bar{\Psi})}\, {}_aD^{\gamma}_{x_\rho}\delta\bar{\Psi}}_{eighhthh} \\ + \underbrace{\frac{\partial L}{\partial ({}_aD^{\gamma}_{x_\rho}A_\mu)}\, {}_{x_\rho}D^{\beta}_{a}\delta A_\mu}_{ninthh} \end{array} \right\} d^4x = 0 \quad (18)$$

Integrating (fourth, fifth, sixth, seventh, eighth, ninth) terms by parts give

$$\delta j = \int \left\{ \begin{array}{c} \left[\frac{\partial L}{\partial \Psi} - {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_\rho}\Psi\right)} - {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_{x_\rho}D^{\beta}_{a}\Psi\right)}\right]\delta\Psi \\ + \\ \left[\frac{\partial L}{\partial \bar{\Psi}} - {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_\rho}\bar{\Psi}\right)} - {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_{x_\rho}D^{\beta}_{a}\bar{\Psi}\right)}\right]\delta\bar{\Psi} \\ + \\ \left[\frac{\partial L}{\partial A_\mu} - {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial({}_aD^{\gamma}_{x_\rho}A_\mu)} - {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial({}_aD^{\gamma}_{x_\rho}A_\mu)}\right]\delta A_\mu \end{array} \right\} d^4x \quad (19)$$

$$= 0 \quad (19)$$

Where

$$ {}_{x_\mu}D^{\beta}_{b} = -{}_aD^{\beta}_{x_\mu} \quad , {}_aD^{\gamma}_{x_\mu} = -{}_{x_\mu}D^{\gamma}_{b} \quad (20)$$

So

$$\delta j = \int \left\{ \begin{array}{l} \left[\frac{\partial L}{\partial \Psi} + {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_aD^{\beta}_{x_\rho}\Psi\right)} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_{x_\rho}D^{\gamma}_{a}\Psi\right)}\right]\delta\Psi + \\ \left[\frac{\partial L}{\partial \bar{\Psi}} + {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_aD^{\beta}_{x_\rho}\bar{\Psi}\right)} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_{x_\rho}D^{\gamma}_{a}\bar{\Psi}\right)}\right]\delta\bar{\Psi} + \\ \left[\frac{\partial L}{\partial A_\mu} + {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_aD^{\beta}_{x_\rho}A_\mu\right)} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial({}_{x_\rho}D^{\gamma}_{a}A_\mu)}\right]\delta A_\mu \end{array} \right\} d^4x = 0 \quad (21)$$

This lead to Euler – Lagrange equations

$$\frac{\partial L}{\partial \Psi} + {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_aD^{\beta}_{x_\rho}\Psi\right)} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_{x_\rho}D^{\gamma}_{a}\Psi\right)} = 0 \quad (22)$$

$$\frac{\partial L}{\partial \bar{\Psi}} + {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_aD^{\beta}_{x_\rho}\bar{\Psi}\right)} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_{x_\rho}D^{\gamma}_{a}\bar{\Psi}\right)} = 0 \quad (23)$$

$$\frac{\partial L}{\partial A_\mu} + {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_aD^{\beta}_{x_\rho}A_\mu\right)} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial({}_{x_\rho}D^{\gamma}_{a}A_\mu)} = 0 \quad (24)$$

$$\frac{\partial L}{\partial \emptyset} + {}_{x_\rho}D^{\beta}_{a}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_\rho}\emptyset\right)} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_{x_\rho}D^{\gamma}_{a}\emptyset\right)} = 0 \quad (25)$$

Note that for fractional calculus of variation problems, the resulting Euler – Lagrange equation contains both the LRLFD and RRLFD.

For $\alpha = \beta = 1$, we have ${}_aD^{\alpha}_{x_v} = \partial_\mu$, ${}_{x_v}D^{\beta}_{b} = -\partial_\mu$ and the equations (22-25) reduce to standard Euler – Lagrange equations.

$$\frac{\partial L}{\partial \Psi} - \partial_\gamma \frac{\partial L}{\partial(\partial_\gamma \Psi)} = 0 \quad (26)$$

$$\frac{\partial L}{\partial \bar{\Psi}} - \partial_\eta \frac{\partial L}{\partial(\partial_\eta \bar{\Psi})} = 0 \quad (27)$$

$$\frac{\partial L}{\partial A^\mu} - \partial_v \frac{\partial L}{\partial(\partial^v A^\mu)} = 0 \quad (28)$$

Let us start with the definition of fractional Yang-Mills Lagrangian density and use the generalization formula of Euler – Lagrange equation to obtain the equations of motion from Yang-Mills Lagrangian density.

$$\mathcal{L} = -\frac{1}{2}\begin{bmatrix} -{}_aD^{\alpha}_{t}A^j\, {}_aD^{\alpha}_{t}A^j + {}_aD^{\alpha}_{t}A^j\, {}_aD^{\alpha}_{x^j}\emptyset \\ -{}_aD^{\alpha}_{x^i}\emptyset\, {}_aD^{\alpha}_{x^i}\emptyset + {}_aD^{\alpha}_{x^i}\emptyset\, {}_aD^{\alpha}_{t}A^i + \\ {}_aD^{\alpha}_{x^i}A^j\, {}_aD^{\alpha}_{x^i}A^j - {}_aD^{\alpha}_{x^i}A^j\, {}_aD^{\alpha}_{x^j}A^i \end{bmatrix} + \begin{bmatrix} i\gamma^0\bar{\Psi}\, {}_aD^{\alpha}_{t}\Psi \\ + i\gamma^i\bar{\Psi}\, {}_aD^{\alpha}_{x_i}\Psi \\ -e\gamma^0\phi\Psi\bar{\Psi} \\ -e\gamma^i A_i\Psi\bar{\Psi} - m\Psi\bar{\Psi} \end{bmatrix} \quad (29)$$

For adjoint field ($\bar{\Psi}$, Ψ , A_μ) equations of motion become

$$\frac{\partial L}{\partial \bar{\Psi}} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_\rho}\bar{\Psi}\right)} = 0 \quad (30)$$

$$\frac{\partial L}{\partial \Psi} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_\rho}\Psi\right)} = 0 \quad (31)$$

$$\frac{\partial L}{\partial A_\mu} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial({}_aD^{\gamma}_{x_\rho}A_\mu)} = 0 \quad (32)$$

$$\frac{\partial L}{\partial \emptyset} + {}_aD^{\gamma}_{x_\rho}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_\rho}\emptyset\right)} = 0 \quad (33)$$

So equations of motion become:

$$\begin{cases} \frac{\partial L}{\partial \bar{\Psi}} + {}_tD^{\gamma}_{a}\frac{\partial L}{\partial({}_aD^{\gamma}_{t}\bar{\Psi})} + {}_{x_j}D^{\gamma}_{a}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_j}\bar{\Psi}\right)} = 0 \\ \frac{\partial L}{\partial \bar{\Psi}} - {}_aD^{\gamma}_{t}\frac{\partial L}{\partial({}_aD^{\gamma}_{t}\bar{\Psi})} - {}_aD^{\gamma}_{x\,j}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_j}\bar{\Psi}\right)} = 0 \end{cases} \quad (34)$$

$$\begin{cases} \frac{\partial L}{\partial \Psi} + {}_tD^{\gamma}_{a}\frac{\partial L}{\partial({}_aD^{\gamma}_{t}\Psi)} + {}_{x_j}D^{\gamma}_{a}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_j}\Psi\right)} = 0 \\ \frac{\partial L}{\partial \Psi} - {}_aD^{\gamma}_{t}\frac{\partial L}{\partial({}_aD^{\gamma}_{t}\Psi)} - {}_aD^{\gamma}_{x\,j}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_j}\Psi\right)} = 0 \end{cases} \quad (35)$$

And

$$\begin{cases} \frac{\partial L}{\partial A_\mu} + {}_tD^{\gamma}_{a}\frac{\partial L}{\partial({}_aD^{\gamma}_{t}A_\mu)} + {}_{x_j}D^{\gamma}_{a}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_j}A_\mu\right)} = 0 \\ \frac{\partial L}{\partial A_\mu} - {}_aD^{\gamma}_{t}\frac{\partial L}{\partial({}_aD^{\gamma}_{t}A_\mu)} - {}_aD^{\gamma}_{x\,j}\frac{\partial L}{\partial\left({}_aD^{\gamma}_{x_j}A_\mu\right)} = 0 \end{cases} \quad (36)$$

$$\begin{cases} \frac{\partial L}{\partial \emptyset} + {}_tD_a^{\gamma} \frac{\partial L}{\partial({}_aD_t^{\gamma}\emptyset)} + {}_{x_j}D_a^{\gamma} \frac{\partial L}{\partial\left({}_aD_{x_j}^{\gamma}\emptyset\right)} = 0 \\ \\ \frac{\partial L}{\partial \emptyset} - {}_aD_t^{\gamma} \frac{\partial L}{\partial({}_aD_t^{\gamma}\emptyset)} - {}_aD_{x\,j}^{\gamma} \frac{\partial L}{\partial\left({}_aD_{x_j}^{\gamma}\emptyset\right)} = 0 \end{cases} \quad (37)$$

And differentiating Yang- Mills Lagrangian density with respect to the adjoint fields ($\overline{\Psi}$, Ψ , A_μ) we obtain

$$\frac{\partial L}{\partial \overline{\Psi}} = i\gamma^0\, {}_aD_t^{\alpha}\Psi + i\gamma^i\, {}_aD_{x_i}^{\alpha}\Psi - e\gamma^i A_i\overline{\Psi} - e\gamma^0\phi\overline{\Psi} - m\overline{\Psi} \quad (38)$$

$$\frac{\partial L}{\partial \Psi} = -e\gamma^i A_i\Psi - e\gamma^0\phi\Psi - m\Psi \quad (39)$$

$$\frac{\partial L}{\partial A_i} = -e\gamma^i\Psi\overline{\Psi} \quad (40)$$

$$\frac{\partial L}{\partial \emptyset} = -e\gamma^0\Psi\overline{\Psi} \quad (41)$$

$$\frac{\partial L}{\partial A^i} = 0 \quad (42)$$

And differentiating the Yang-Mills Lagrangian density with respect to the time derivative fields (${}_aD_t^{\alpha}\Psi, {}_aD_t^{\alpha}\overline{\Psi}, {}_aD_t^{\gamma} A_\mu$) we obtain

$$\frac{\partial L}{\partial({}_aD_t^{\alpha}\Psi)} = i\gamma^0\overline{\Psi} \quad (43)$$

$$\frac{\partial L}{\partial({}_aD_t^{\alpha}\overline{\Psi})} = 0 \quad (44)$$

$$\frac{\partial L}{\partial({}_aD_t^{\alpha}A^i)} = -\frac{1}{2}\left({}_aD_{x^i}^{\alpha}\emptyset\right) \quad (45)$$

$$\frac{\partial L}{\partial({}_aD_t^{\alpha}A^j)} = -\frac{1}{2}\left(2\,{}_aD_t^{\alpha}A^j - {}_aD_{x^i}^{\alpha}\phi\right) \quad (46)$$

$$\frac{\partial L}{\partial({}_aD_t^{\gamma}\emptyset)} = 0 \quad (47)$$

The differential Yang-Mills Lagrangian density with respect to the space derivative field (${}_aD_{x_j}^{\alpha}\Psi, {}_aD_{x_j}^{\alpha}\overline{\Psi}, {}_aD_{x_j}^{\gamma} A_\mu$), gives:

$$\frac{\partial L}{\partial\left({}_aD_{x_j}^{\gamma}\Psi\right)} = i\gamma^i\overline{\Psi} \quad (48)$$

$$\frac{\partial L}{\partial\left({}_aD_{x_j}^{\gamma}\overline{\Psi}\right)} = 0 \quad (49)$$

$$\frac{\partial L}{\partial\left({}_aD_{x^j}^{\gamma}A^i\right)} = -\frac{1}{2}\left({}_aD_{x^i}^{\alpha}A^j\right) \quad (50)$$

$$\frac{\partial L}{\partial\left({}_aD_{x^i}^{\gamma}A^j\right)} = -\frac{1}{2}\left({}_aD_{x^i}^{\alpha}A^j - {}_aD_{x^j}^{\alpha}A^i\right) \quad (51)$$

$$\frac{\partial L}{\partial\left({}_aD_{x_j}^{\gamma}\emptyset\right)} = \frac{1}{2}\left(-2\,{}_aD_{x^i}^{\alpha}\emptyset - {}_aD_t^{\alpha}A^i\right) \quad (52)$$

Substituting equations (38, 44, and 49) in equation (34) we get

$$\left[i\gamma^0\, {}_aD_t^{\alpha}\Psi + i\gamma^i\, {}_aD_{x_i}^{\alpha}\Psi - e\gamma^i A_i\Psi - e\gamma^0\phi\Psi - m\Psi\right] = 0 \quad (53)$$

Or

$$\left[i\gamma^\mu\, {}_aD_{x^\mu}^{\alpha}\Psi - e\gamma^\mu A_\mu\Psi - m\Psi\right] = 0 \quad (54)$$

And substituting equations (39, 43, 48) in equation (36) we get

$$\left[-e\gamma^i A_i\overline{\Psi} - e\gamma^0\phi\overline{\Psi} + i\gamma^i\, {}_{x_j}D_a^{\gamma}\overline{\Psi} + i\gamma^0\, {}_tD_a^{\gamma}\overline{\Psi} - m\overline{\Psi}\right] = 0 \quad (55)$$

Or

$$\left[i\gamma^\mu\, {}_{x_\mu}D_a^{\gamma}\overline{\Psi} + e\gamma^\mu A_\mu\overline{\Psi} + m\overline{\Psi}\right] = 0 \quad (56)$$

And substituting equations (40, 41, 42, 45, 46, 47, 50, 51, 52) in equations (36) and (37) we get

$$\Big[-e\gamma^i\Psi\overline{\Psi} - \frac{1}{2}\,{}_aD_t^{\alpha}\left({}_aD_{x^i}^{\alpha}\emptyset\right) - \frac{1}{2}\,{}_aD_{x_j}^{\gamma}\left(2\,{}_aD_t^{\alpha}A^j - {}_aD_{x^i}^{\alpha}\phi\right) - \frac{1}{2}\,{}_aD_{x^j}^{\gamma}\left({}_aD_{x^i}^{\alpha}A^j\right) - \frac{1}{2}\,{}_aD_{x^i}^{\gamma}\left({}_aD_{x^i}^{\alpha}A^j - {}_aD_{x^j}^{\alpha}A^i\right) + \frac{1}{2}\,{}_aD_{x_j}^{\gamma}\left(-{}_aD_{x^i}^{\alpha}\emptyset - {}_aD_t^{\alpha}A^i\right)\Big] = 0 \quad (57)$$

Or

$$\left[e\gamma^\mu\Psi\overline{\Psi} + {}_aD_{x_\nu}^{\alpha}\left({}_aD_{x_\mu}^{\alpha}A^\nu - {}_aD_{x_\nu}^{\alpha}A^\mu\right)\right] = 0 \quad (58)$$

For $\alpha = 1$, we have ${}_aD_{x^\mu}^{\alpha} = \partial_\mu$ the equations (54, 56, and 58) then reduce to standard Euler – Lagrange equations:

$$\left[i\gamma^\mu\, {}_aD_{x^\mu}^{\alpha}\Psi - e\gamma^\mu A_\mu\Psi - m\Psi\right] = 0$$

Or

$$\left[i\gamma^\mu \frac{\partial}{\partial x^\mu}\Psi - e\gamma^\mu A_\mu\Psi - m\Psi\right] = 0$$

And

$$\left[i\gamma^\mu\, {}_{x_j}D_a^{\gamma}\overline{\Psi} + e\gamma^\mu A_\mu\overline{\Psi} + m\overline{\Psi}\right] = 0$$

Or

$$\left[i\gamma^\mu \frac{\partial}{\partial x^\mu}\overline{\Psi} + e\gamma^\mu A_\mu\overline{\Psi} + m\overline{\Psi}\right] = 0$$

And

$$\left[e\gamma^\mu\Psi\overline{\Psi} + {}_aD_{x_\nu}^{\alpha}\left({}_aD_{x_\mu}^{\alpha}A^\nu - {}_aD_{x_\nu}^{\alpha}A^\mu\right)\right] = 0$$

Or

$$\left[\partial_\nu F_{\mu\nu} + e\gamma^\mu\Psi\overline{\Psi}\right] = 0$$

3.3 Fractional Hamiltonian Formulation

To construct the fractional Hamiltonian equation within Riemann – Liouville fractional derivative from fractional Yang-Mills Lagrangian density, we consider the Lagrangian depending on fractional time derivatives of coordinates in the form:

$$\mathcal{L} = \mathcal{L}\left(\Psi, {}_aD_t^{\alpha}\Psi, {}_aD_{x_j}^{\alpha}\Psi, \overline{\Psi}, {}_aD_t^{\alpha}\overline{\Psi}, {}_aD_{x_j}^{\alpha}\overline{\Psi}, \emptyset, A^i, A^j, {}_aD_t^{\alpha}A^j, {}_aD_t^{\alpha}A^i, {}_aD_t^{\alpha}\emptyset, {}_aD_{x_i}^{\alpha}A^j, {}_aD_{x^j}^{\alpha}A^i, {}_aD_{x^i}^{\alpha}\emptyset, t\right) \quad (59)$$

The Hamiltonian depending on the fractional time derivatives reads as

$$\mathcal{H} = \pi\, {}_aD_t^{\alpha}\Psi + \bar{\pi}\, {}_aD_t^{\alpha}\overline{\Psi} + \pi_{\alpha_{A^i}}\, {}_aD_t^{\alpha}A^i + \pi_{\alpha_{A^j}}\, {}_aD_t^{\alpha}A^j -$$

$\mathcal{L}\,(\Psi,\,{}_aD_t^\alpha\Psi,\,{}_aD_{x_j}^\alpha\Psi,\overline{\Psi},\,{}_aD_t^\alpha\overline{\Psi},\,{}_aD_{x_j}^\alpha\overline{\Psi},\emptyset,A^i,A^j,\,{}_aD_t^\alpha A^j,$
${}_aD_t^\alpha A^i,\,{}_aD_t^\alpha\emptyset\,,\,{}_aD_{x_i}^\alpha A^j,\,{}_aD_{x^j}^\alpha A^i,\,{}_aD_{x^i}^\alpha\emptyset\,,\,t)$ (60)

Where

$\mathcal{H}=\mathcal{H}\,(\pi\,,\,\Psi,\bar{\pi},\overline{\Psi},\,{}_aD_{x_j}^\alpha\Psi,\,{}_aD_{x_j}^\alpha\overline{\Psi}\,,\emptyset\,,A^i\,,A^j,t\,,\;\pi_{\alpha_{A^i}}$
$,\pi_{\alpha_{A^j}},\,{}_aD_{x^i}^\alpha\emptyset,\;\;{}_aD_{x^i}^\alpha A^j,\;\;\;{}_aD_{x^j}^\alpha A^i)$ (61)

Thus, the total differential of Hamiltonian function reads as:

$$dH = \begin{cases} \dfrac{\partial H}{\partial\Psi}d\Psi+\dfrac{\partial H}{\partial\pi}d\pi+\dfrac{\partial H}{\partial\left({}_aD_{x_j}^\alpha\Psi\right)}d\left({}_aD_{x_j}^\alpha\Psi\right) \\ +\dfrac{\partial H}{\partial\overline{\Psi}}d\overline{\Psi}+\dfrac{\partial H}{\partial\bar{\pi}}d\bar{\pi}+\dfrac{\partial H}{\partial\left({}_aD_{x_j}^\alpha\overline{\Psi}\right)}d\left({}_aD_{x_j}^\alpha\overline{\Psi}\right) \\ +\dfrac{\partial H}{\partial\pi_{\alpha_{A^j}}}d\pi_{\alpha_{A^j}}+\dfrac{\partial H}{\partial\pi_{\alpha_{A^i}}}d\pi_{\alpha_{A^i}}+\dfrac{\partial H}{\partial A^j}dA^j \\ +\dfrac{\partial H}{\partial A^i}dA^i+\dfrac{\partial H}{\partial\emptyset}d\emptyset+\dfrac{\partial H}{\partial t}dt+ \\ \dfrac{\partial H}{\partial\left({}_aD_{x^i}^\alpha\emptyset\right)}d\left({}_aD_{x^i}^\alpha\emptyset\right)+\dfrac{\partial H}{\partial\left({}_aD_{x^i}^\alpha A^j\right)}\,d\left({}_aD_{x^i}^\alpha A^j\right) \\ +\dfrac{\partial H}{\partial({}_aD_{x^j}^\alpha A^i)}d\left({}_aD_{x^j}^\alpha A^i\right) \end{cases} \tag{62}$$

And

$$dH = \begin{cases} \pi d({}_aD_t^\alpha\Psi)+{}_aD_t^\alpha\Psi d(\pi)+\bar{\pi}d({}_aD_t^\alpha\overline{\Psi}) \\ +{}_aD_t^\alpha\overline{\Psi}d(\bar{\pi})+d\pi_{\alpha_{A^i}}({}_aD_t^\alpha A^i)\;+\pi_{\alpha_{A^j}}d({}_aD_t^\alpha A^j) \\ -\dfrac{\partial L}{\partial\left({}_aD_t^\alpha\;A^i\right)}d\left({}_aD_t^\alpha\;A^i\right)-\dfrac{\partial L}{\partial({}_aD_t^\alpha A^j)}d({}_aD_t^\alpha A^j) \\ -\dfrac{\partial L}{\partial\left({}_aD_t^\alpha\overline{\emptyset}\right)}d({}_aD_t^\alpha\emptyset)+\pi_{\alpha_{A^i}}d({}_aD_t^\alpha A^i)+d\pi_{\alpha_{A^j}}({}_aD_t^\alpha A^j) \\ -\dfrac{\partial L}{\partial t}dt-\dfrac{\partial L}{\partial\Psi}d\Psi-\dfrac{\partial L}{\partial({}_aD_t^\alpha\Psi)}d({}_aD_t^\alpha\Psi) \\ -\dfrac{\partial L}{\partial\left({}_aD_{x_j}^\alpha\Psi\right)}d\left({}_aD_{x_j}^\alpha\Psi\right)-\dfrac{\partial L}{\partial\overline{\Psi}}d\overline{\Psi}-\dfrac{\partial L}{\partial\left({}_aD_{x_j}^\alpha\overline{\Psi}\right)}d\left({}_aD_{x_j}^\alpha\overline{\Psi}\right) \\ -\dfrac{\partial L}{\partial({}_aD_t^\alpha\overline{\Psi})}d({}_aD_t^\alpha\overline{\Psi})-\dfrac{\partial L}{\partial A^j}dA^j-\dfrac{\partial L}{\partial A^i}dA^i \\ -\dfrac{\partial L}{\partial\left({}_aD_{x^j}^\alpha A^i\right)}d\left({}_aD_{x^j}^\alpha A^i\right)-\dfrac{\partial L}{\partial\emptyset}d\emptyset \\ -\dfrac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^j\right)}d\left({}_aD_{x^i}^\alpha A^j\right)-\dfrac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\emptyset\right)}d\left({}_aD_{x^i}^\alpha\emptyset\right) \end{cases} \tag{63}$$

After substituting the values of conjugate momenta in equation (63) and then comparing equation (62) with equation (63) we get the following Hamilton's equation of motion:

$$\begin{cases} \dfrac{\partial H}{\partial t}=-\dfrac{\partial L}{\partial t} \\ \dfrac{\partial H}{\partial\pi_{\alpha_{A^j}}}={}_aD_t^\alpha A^j \\ \pi_{\alpha_{A^j}}={}_aD_t^\alpha A^j \end{cases} \tag{64}$$

$$\begin{cases} \dfrac{\partial H}{\partial\left({}_aD_{x^i}^\alpha\emptyset\right)}=-\dfrac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\emptyset\right)} \\ \dfrac{\partial H}{\partial\left({}_aD_{x^i}^\alpha A^j\right)}=-\dfrac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^j\right)} \\ \dfrac{\partial H}{\partial\left({}_aD_{x^j}^\alpha A^i\right)}=-\dfrac{\partial L}{\partial\left({}_aD_{x^j}^\alpha A^i\right)} \end{cases} \tag{65}$$

$$\begin{cases} \dfrac{\partial H}{\partial\emptyset}=-{}_tD_b^\alpha\dfrac{\partial L}{\partial({}_aD_t^\alpha\emptyset)}-{}_{x^i}D_b^\alpha\dfrac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\emptyset\right)} \\ \dfrac{\partial H}{\partial A^i}=-{}_tD_b^\alpha\dfrac{\partial L}{\partial({}_aD_t^\alpha A^i)}-{}_{x^i}D_b^\alpha\dfrac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^I\right)} \\ \dfrac{\partial H}{\partial A^j}=-{}_tD_b^\alpha\dfrac{\partial L}{\partial({}_aD_t^\alpha A^j)}-{}_{x^i}D_b^\alpha\dfrac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^j\right)} \end{cases} \tag{66}$$

$$\begin{cases} \dfrac{\partial H}{\partial\Psi}=-{}_aD_t^\alpha\dfrac{\partial L}{\partial({}_aD_t^\mu\Psi)}-{}_aD_{x_j}^\alpha\dfrac{\partial L}{\partial({}_aD_{x_j}^\mu\Psi)} \\ \dfrac{\partial H}{\partial\Psi}=-{}_{x_j}D_b^\alpha\dfrac{\partial L}{\partial\left({}_aD_{x_j}^\alpha\Psi\right)} \\ \dfrac{\partial H}{\partial\pi}={}_aD_t^\alpha\varphi \end{cases} \tag{67}$$

$$\begin{cases} \dfrac{\partial H}{\partial\overline{\Psi}}=-{}_aD_t^\alpha\dfrac{\partial L}{\partial({}_aD_t^\mu\overline{\Psi})}-{}_aD_{x_j}^\alpha\dfrac{\partial L}{\partial({}_aD_{x_j}^\mu\overline{\Psi})} \\ \dfrac{\partial H}{\partial\overline{\Psi}}=-{}_{x_j}D_b^\alpha\dfrac{\partial L}{\partial\left({}_aD_{x_j}^\alpha\overline{\Psi}\right)} \\ \dfrac{\partial H}{\partial\bar{\pi}}={}_aD_t^\alpha\overline{\Psi} \end{cases} \tag{68}$$

As an example consider the Yang-Mills Lagrangian vector density

$$\mathcal{L}=-\frac{1}{2}\begin{bmatrix} -{}_aD_t^\alpha A^j\,{}_aD_t^\alpha A^j+{}_aD_t^\alpha A^j\,{}_aD_{x^j}^\alpha\emptyset \\ -{}_aD_{x^i}^\alpha\emptyset\,{}_aD_{x^i}^\alpha\emptyset+\;{}_aD_{x^i}^\alpha\emptyset\,{}_aD_t^\alpha A^i+ \\ {}_aD_{x^i}^\alpha A^j\,{}_aD_{x^i}^\alpha A^j-{}_aD_{x^i}^\alpha A^j\,{}_aD_{x^j}^\alpha A^i \end{bmatrix}+\begin{bmatrix} i\gamma^0\overline{\Psi}\,{}_aD_t^\alpha\Psi \\ +i\gamma^i\overline{\Psi}\,{}_aD_{x_i}^\alpha\Psi \\ -e\gamma^0\phi\Psi\overline{\Psi}- \\ e\gamma^iA_i\Psi\overline{\Psi}-m\Psi\overline{\Psi} \end{bmatrix} \tag{69}$$

But the Euler – Lagrange equation is

$$\frac{\partial L}{\partial\Psi}-{}_aD_{x_\mu}^\alpha\frac{\partial L}{\partial({}_aD_{x_\mu}^\alpha\Psi)}=0 \tag{70}$$

Or

$$\frac{\partial L}{\partial\Psi}={}_aD_t^\alpha\frac{\partial L}{\partial({}_aD_t^\alpha\Psi)}+{}_aD_{x_j}^\alpha\frac{\partial L}{\partial\left({}_aD_{x_j}^\alpha\Psi\right)} \tag{71}$$

And

$$\frac{\partial L}{\partial \overline{\Psi}} - {}_aD^{\alpha}_{x_\mu}\frac{\partial L}{\partial({}_aD^{\alpha}_{x_\mu}\overline{\Psi})} = 0 \quad (72)$$

Or

$$\frac{\partial L}{\partial \overline{\Psi}} = {}_aD^{\alpha}_{t}\frac{\partial L}{\partial({}_aD^{\alpha}_{t}\overline{\Psi})} + {}_aD^{\alpha}_{x_j}\frac{\partial L}{\partial({}_aD^{\alpha}_{x_j}\overline{\Psi})} \quad (73)$$

And

$$\frac{\partial L}{\partial A_\mu} - {}_aD^{\gamma}_{x_\mu}\frac{\partial L}{\partial({}_aD^{\gamma}_{x_\mu}A_\mu)} = 0 \quad (74)$$

$$\frac{\partial L}{\partial A_\mu} = {}_aD^{\alpha}_{t}\frac{\partial L}{\partial({}_aD^{\alpha}_{t}A_\mu)} + {}_aD^{\alpha}_{x_j}\frac{\partial L}{\partial({}_aD^{\alpha}_{x_j}A_\mu)} = 0 \quad (75)$$

So Euler – Lagrange equations become

$$\begin{bmatrix} -e\gamma^0\phi\overline{\Psi} - e\gamma^i A_i\overline{\Psi} - m\overline{\Psi} - \\ i\gamma^0{}_aD^{\gamma}_t\overline{\Psi} - i\gamma^i{}_aD^{\gamma}_{x_j}\overline{\Psi}\end{bmatrix} = 0$$

Or

$$\left[i\gamma^{\mu}{}_{x_j}D^{\gamma}_a\overline{\Psi} + e\gamma^{\mu}A_\mu\overline{\Psi} + m\overline{\Psi}\right] = 0 \quad (76)$$

And

$$\begin{bmatrix} i\gamma^0{}_aD^{\alpha}_t\Psi + i\gamma^i{}_aD^{\alpha}_{x_i}\Psi - e\gamma^0\phi\Psi \\ - e\gamma^i A_i\Psi - m\Psi\end{bmatrix} = 0 \quad (77)$$

Or

$$\left[i\gamma^{\mu}{}_aD^{\alpha}_{x^\mu}\Psi - e\gamma^{\mu}A_\mu\Psi - m\Psi\right] = 0 \quad (78)$$

And

$$\begin{bmatrix} -e\gamma^i\Psi\overline{\Psi} - e\gamma^i\Psi\overline{\Psi} - \frac{1}{2}{}_aD^{\alpha}_t\left({}_aD^{\alpha}_{x^i}\emptyset\right) \\ -\frac{1}{2}{}_aD^{\gamma}_{x_j}\left(2{}_aD^{\alpha}_tA^j - {}_aD^{\alpha}_{x^i}\phi\right) \\ -\frac{1}{2}{}_aD^{\gamma}_{x^j}\left({}_aD^{\alpha}_{x^i}A^j\right) - \frac{1}{2}{}_aD^{\gamma}_{x^i}\left({}_aD^{\alpha}_{x^i}A^j - {}_aD^{\alpha}_{x^j}A^i\right)\end{bmatrix} = 0 \quad (79)$$

Or

$$\left[ie\gamma^{\mu}\Psi\overline{\Psi} + {}_aD^{\alpha}_{x_\nu}\left({}_aD^{\alpha}_{x_\mu}A^{\nu} - {}_aD^{\alpha}_{x_\nu}A^{\mu}\right)\right] = 0 \quad (80)$$

As $\alpha = 1$ we obtain the following equations

$$\left(e\gamma^{\mu}A_\mu + m + i\gamma^{\mu}\partial_\mu\right)\overline{\Psi} = 0$$

$$\left(i\gamma^{\mu}\partial_\mu - e\gamma^{\mu}A_\mu - m\right)\Psi = 0$$

$$\left[\partial_\nu F_{\mu\nu} + e\gamma^{\mu}\Psi\overline{\Psi}\right] = 0$$

Now the fractional Hamilton's equation of motion is given by

$$\mathcal{H} = \pi{}_aD^{\alpha}_t\Psi + \bar{\pi}{}_aD^{\alpha}_t\overline{\Psi} + \pi^{A^i}{}_aD^{\alpha}_tA_i - \mathcal{L} \quad (81)$$

By substituting the conjugate fields in fractional form we get

$$\pi = \frac{\partial L}{\partial({}_aD^{\alpha}_t\Psi)}$$

$$\bar{\pi} = \frac{\partial L}{\partial({}_aD^{\alpha}_t\overline{\Psi})}$$

$$\pi^i = \frac{\partial L}{\partial({}_aD^{\alpha}_tA_i)}$$

Where the Lagrangian density in fractional form is given by

$$\mathcal{L} = -\frac{1}{2}\begin{bmatrix} -{}_aD^{\alpha}_tA^j{}_aD^{\alpha}_tA^j + {}_aD^{\alpha}_tA^j{}_aD^{\alpha}_{x^j}\emptyset \\ -{}_aD^{\alpha}_{x^i}\emptyset{}_aD^{\alpha}_{x^i}\emptyset + {}_aD^{\alpha}_{x^i}\emptyset{}_aD^{\alpha}_tA^i + \\ {}_aD^{\alpha}_{x^i}A^j{}_aD^{\alpha}_{x^i}A^j - {}_aD^{\alpha}_{x^i}A^j{}_aD^{\alpha}_{x^j}A^i\end{bmatrix} + \begin{bmatrix} i\gamma^0\overline{\Psi}{}_aD^{\alpha}_t\Psi \\ + i\gamma^i\overline{\Psi}{}_aD^{\alpha}_{x_i}\Psi \\ -e\gamma^0\phi\Psi\overline{\Psi} - \\ e\gamma^iA_i\Psi\overline{\Psi} - m\Psi\overline{\Psi}\end{bmatrix} \quad (82)$$

So the Hamiltonian becomes

$$\mathcal{H} = \pi{}_aD^{\alpha}_t\Psi + \bar{\pi}{}_aD^{\alpha}_t\overline{\Psi} + \pi^{A^i}{}_aD^{\alpha}_tA_i + \frac{1}{2}\begin{bmatrix} -{}_aD^{\alpha}_tA^j{}_aD^{\alpha}_tA^j + {}_aD^{\alpha}_tA^j{}_aD^{\alpha}_{x^j}\emptyset \\ -{}_aD^{\alpha}_{x^i}\emptyset{}_aD^{\alpha}_{x^i}\emptyset + {}_aD^{\alpha}_{x^i}\emptyset{}_aD^{\alpha}_tA^i + \\ {}_aD^{\alpha}_{x^i}A^j{}_aD^{\alpha}_{x^i}A^j - {}_aD^{\alpha}_{x^i}A^j{}_aD^{\alpha}_{x^j}A^i\end{bmatrix} - \begin{bmatrix} i\gamma^0\overline{\Psi}{}_aD^{\alpha}_t\Psi \\ + i\gamma^i\overline{\Psi}{}_aD^{\alpha}_{x_i}\Psi \\ -e\gamma^0\phi\Psi\overline{\Psi} - \\ e\gamma^iA_i\Psi\overline{\Psi} - m\Psi\overline{\Psi}\end{bmatrix} \quad (83)$$

Now the equations of motion are given by

$$\begin{cases} \frac{\partial H}{\partial \Psi} = -{}_aD^{\alpha}_t\frac{\partial L}{\partial({}_aD^{\alpha}_t\Psi)} - {}_aD^{\alpha}_{x_j}\frac{\partial L}{\partial({}_aD^{\alpha}_{x_j}\Psi)} \\ \frac{\partial H}{\partial \Psi} = -{}_{x_j}D^{\alpha}_b\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x_j}\Psi\right)} \\ \frac{\partial H}{\partial \pi} = {}_aD^{\alpha}_t\Psi\end{cases} \quad (84)$$

So

$$\frac{\partial H}{\partial \Psi} = -{}_aD^{\alpha}_t\frac{\partial L}{\partial({}_aD^{\alpha}_t\Psi)} - {}_aD^{\alpha}_{x_j}\frac{\partial L}{\partial({}_aD^{\alpha}_{x_j}\Psi)} \quad (85)$$

$$-e\gamma^0\phi\overline{\Psi} - e\gamma^iA_i\overline{\Psi} - m\overline{\Psi} = -{}_aD^{\alpha}_t{}_aD^{\alpha}_t\Psi - {}_aD^{\alpha}_{x_j}{}_aD^{\alpha}_{x_j}\Psi \quad (86)$$

$$-e\gamma^0\phi\overline{\Psi} - e\gamma^iA_i\overline{\Psi} - m\overline{\Psi} = +{}_aD^{\alpha}_t{}_a\overline{\Psi} + {}_aD^{\alpha}_{x_j}\overline{\Psi} \quad (87)$$

So the Euler –Lagrange equation becomes

$$\left[i\gamma^{\mu}{}_{x_j}D^{\gamma}_a\overline{\Psi} + e\gamma^{\mu}A_\mu\overline{\Psi} + m\overline{\Psi}\right] = 0 \quad (88)$$

And

$$\begin{cases} \frac{\partial H}{\partial \overline{\Psi}} = -{}_aD^{\alpha}_t\frac{\partial L}{\partial({}_aD^{\mu}_t\overline{\Psi})} - {}_aD^{\alpha}_{x_j}\frac{\partial L}{\partial({}_aD^{\mu}_{x_j}\overline{\Psi})} \\ \frac{\partial H}{\partial \overline{\Psi}} = -{}_{x_j}D^{\alpha}_b\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x_j}\overline{\Psi}\right)}d\left({}_aD^{\alpha}_{x^i}\overline{\Psi}\right) \\ \frac{\partial H}{\partial \bar{\pi}} = {}_aD^{\alpha}_t\overline{\Psi}\end{cases} \quad (89)$$

So

$$\begin{pmatrix} i\gamma^0{}_aD^{\alpha}_t\Psi + i\gamma^i{}_aD^{\alpha}_{x_i}\Psi - e\gamma^0\phi\Psi \\ -e\gamma^iA_i\Psi - m\Psi\end{pmatrix} = 0 \quad (90)$$

So the Euler –Lagrange equation becomes

$$\left[i\gamma^{\mu}{}_aD^{\gamma}_{x_j}\Psi - e\gamma^{\mu}A_\mu\Psi - m\Psi\right] = 0 \quad (91)$$

$$\begin{cases} \frac{\partial H}{\partial \emptyset} = -{}_tD^{\alpha}_b\frac{\partial L}{\partial({}_aD^{\alpha}_t\emptyset)} - {}_{x^i}D^{\alpha}_b\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x^i}\emptyset\right)} \\ \frac{\partial H}{\partial A^i} = -{}_tD^{\alpha}_b\frac{\partial L}{\partial({}_aD^{\alpha}_tA^i)} - {}_{x^i}D^{\alpha}_b\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x^i}A^I\right)} \\ \frac{\partial H}{\partial A^j} = -{}_tD^{\alpha}_b\frac{\partial L}{\partial({}_aD^{\alpha}_tA^j)} - {}_{x^i}D^{\alpha}_b\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x^i}A^j\right)}\end{cases} \quad (92)$$

So

$$\frac{\partial H}{\partial \emptyset} = -e\gamma^0\Psi\overline{\Psi}$$

$$-e\gamma^0\Psi\overline{\Psi} = {}_aD^{\alpha}_t\left[-\frac{1}{2}(0)\right] + {}_aD^{\alpha}_{x^i}\left[-\frac{1}{2}\left(2{}_aD^{\alpha}_tA^i - 2{}_aD^{\alpha}_{x^i}\emptyset\right)\right] \quad (93)$$

And

$$\frac{\partial H}{\partial A^i} = -{}_tD^{\alpha}_b\frac{\partial L}{\partial({}_aD^{\alpha}_tA^i)} - {}_{x^i}D^{\alpha}_b\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x^i}A^i\right)}$$

$$-e\gamma^{i}\,\Psi\overline{\Psi} = {}_{a}D_{t}^{\alpha}\left[-\frac{1}{4}\left({}_{a}D_{x^{i}}^{\alpha}\emptyset\right)\right] + {}_{a}D_{x^{j}}^{\alpha}\left[-\frac{1}{4}\left(-{}_{a}D_{x^{i}}^{\alpha}A^{j}\right)\right] \quad (94)$$

And

$$\frac{\partial H}{\partial A^{j}} = -{}_{t}D_{b}^{\alpha}\frac{\partial L}{\partial\left({}_{a}D_{t}^{\alpha}A^{j}\right)} - {}_{x^{i}}D_{b}^{\alpha}\frac{\partial L}{\partial\left({}_{a}D_{x^{i}}^{\alpha}A^{j}\right)}$$

$$0 = \frac{1}{2}\left[-2\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}A^{j} + {}_{a}D_{t}^{\alpha}\,{}_{a}D_{x^{j}}^{\alpha}\emptyset 2\,{}_{a}D_{x^{i}}^{\alpha}\,{}_{a}D_{x^{j}}^{\alpha}A^{j} - 2\,{}_{a}D_{x^{i}}^{\alpha}\,{}_{a}D_{x^{j}}^{\alpha}A^{i}\right] \quad (95)$$

$$-e\gamma^{0}\,\Psi\overline{\Psi} - e\gamma^{i}\,\Psi\overline{\Psi} = \begin{bmatrix} {}_{a}D_{t}^{\alpha}\left({}_{a}D_{x^{i}}^{\alpha}\emptyset + {}_{a}D_{t}^{\alpha}A^{j}\right) + \\ {}_{a}D_{x^{i}}^{\alpha}\left({}_{a}D_{x^{i}}^{\alpha}A^{j} - {}_{a}D_{x^{j}}^{\alpha}A^{i}\right) \end{bmatrix}$$

So the Euler –Lagrange equation becomes

$$\left[\, ie\gamma^{\mu}\Psi\overline{\Psi} + {}_{a}D_{x_{v}}^{\alpha}\left({}_{a}D_{x_{\mu}}^{\alpha}A^{v} - {}_{a}D_{x_{v}}^{\alpha}A^{\mu}\right)\,\right] = 0 \quad (96)$$

As $\alpha = 1$ we obtain the equations of motion

$$\left(e\gamma^{\mu}A_{\mu} + m + i\gamma^{\mu}\partial_{\mu}\right)\overline{\Psi} = 0$$

$$\left(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu} - m\right)\Psi = 0$$

$$\left[\partial_{v}F_{\mu v} + e\gamma^{\mu}\Psi\overline{\Psi}\right] = 0$$

4. CONCLUSION

The Yang-Mills field is reformulated using fractional calculus with left-right Riemann-Liouville fractional derivatives. Both fractional Euler-Lagrange equations and fractional Hamiltonian equations give the same results in the two procedures used in this paper for Yang-Mills Lagrangian density. The classical results are obtained as a particular case of fractional formulation.

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